RESEARCH ARTICLE

Abundant M-Fractional Exact Solutions for STO and (3 + 1)-Dimensional KdV-ZK Equations via (m + (G′/G))-Expansion Method

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ABSTRACT

In the field of applied mathematics, fractional calculus is used to contract with derivative as well as the integration of any power. Different definitions of the fractional derivative have been introduced in the literature. For example, these are some important definitions of fractional derivatives, Riemann-Liouville derivative, Caputo derivative and conformable derivative. Recently the generalization of the conformable derivative has been given as M-fractional conformable. Fractional differential equations (FDEs) "equations involving fractional derivatives" are employed invarious areas of science and engineering and others [1-5] have widely been interested. That's why they have gained many attractions from many researchers. To acquisition, the analytical solutions of the FPDEs is a conspicuous look of scientific research.

Key Words: *Fractional Derivative, Riemann-Liouville derivative, Conformable derivative and numerous scholars*

INTRODUCTION

Consequently, numerous scholars have developed some persuasive methods to acquired approximate and exact solutions for these types of FPDEs. In this investigation, the truncated M-fractional conformable (STO) and (3 + 1)-dimensional KdV-ZK equations are considered. The novel method: (m + G')-expansion method are utilized to extract the exact solitons of the aforesaid model equations. Different definitions of fractional derivatives have appeared in the literature. For example, Reimann-Liouville [1], Caputo derivative and conformable derivative. Despite these two most recent definitions are reported as M-fractional conformable and beta derivatives. Many powerful techniques have been reported in the literature for finding exact solutions, see for example [5-9]. Fortunately, it is possible to establish a traveling wave transformation for a fractional order PDE which can convert it to a nonlinear ordinary differential equation (NODE) that can be easily solved by using a variety of different methods. There are many distinct techniques have been applied to gain the exact solitons of (STO) and (3 + 1)-dimensional KdV-ZK equations like : F-expansion method and Improved - expansion methods are used to obtain the bright, dark 1-soliton and other soliton solutions [10], new extended direct algebraic technique is implemented to gain the number of new type of solitons of the conformable fractional (STO) equation [11], dark and bright optical solitons gained by variable coefficient method [12], dispersive exact wave solutions are observed by modified simplest equation method [13], different optical solitons of the (STO) equation are achieved by applying the extended trial method [14], Extended sinh-Gordon equation expansion scheme has utilized to obtain different types of optical solitons of STO equation [15], extended auxiliary equation scheme is applied to gain the dispersive optical wave solitons of time-fractional (SH) equation along power law non-linearity as well as Kerr Law non-linearity [16], undetermined coefficient method is implemented to gain the distinct kinds of dispersive exact solitons in the presence of several perturbation terms are achieved [17], with the use of tanh- coth integration algorithm dispersive solitons in optical nanofibers are obtained with constraint conditions [18], by using the Sine-Cosine function method, different exact solutions are obtained [19], Sech, Tanh and Csch function techniques are utilized to gain the optical solitons of (STO) equation along Kerr law non-linearity [20]. There are many applications of the Sine-Gordon expansion method and (m + G/G) -expansion method. For instance, with the use of Sine Gordon-expansion scheme, distinct kinds of solitons of the non-linear time-fractional Biological Population equation and the Cahn- Hilliard model have been obtained in [21], hyperbolic and trigonometric functions solitons to the non-linear reaction diffusion equation have achieved [22] etc. Similarly, (m + G/G)-expansion method has been utilized to solve the Pochhammer-Chree equations for bell-shaped, kink-shaped and periodic type solitons of with the help of this method [23], discrete and periodic type solitons of the Ablowitz-Ladik lattice system are found [24] etc.

To find the exact solutions of integrable partial differential equations is the most interesting topic. Therefore, we will solve two integrable model equations namely space-time fractional Sharma Tasso-Olever (STO) and space- time fractional (3+1)-dimensional KdV-ZK equations for a variety of solitons with a novel derivative operator by employing (m + G/G)-expansion method: Abundant M-Fractional Exact Solutions for STO and (3+1)-Dimensional KdV-ZK Equations via (m + G/G)-Expansion Method

2 Description of (m + G′/G)-Expansion method

In this section, the basic concept of the (m + G/G) method are illustrated as bellow:

Step:1

Assume that the general form of NLPDE is expressed as

P (u, ux, ut, uxx, ...) = 0. **(1)**

Suppose the Wave transformation takes the following form**:**

$$u(x, t) = U(\xi),\ \xi = x + v(t) \tag{2}$$

Inserting eq.(2) into eq.(1),we obtain

P (U, U,, U,,, ...) = 0. (3)

Step:2

Suppose that the trial solution of eq.(3) is given by

$$U(\xi) = \sum_{l=-n}^{n} c_l (m+F)^l = c_{-n}(m+F)^{-n} + \dots\dots\dots\dots + c_0 + c_1(m+F) + \dots\dots + c_n(m+F)^n \tag{4}$$

where cn, n = 0, 1, , , , , , , , n and m are nonzero constants.

According to the principles of balance,we find the value of n.In this article we define F as

$$F = \frac{G'(\xi)}{G(\xi)} \tag{5}$$

where G(ξ) satisfies,the second order nonlinear ODE:

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$GG'' = pGG' + qG2 + r(G')2$ (6)

The Cole-Hopf transformation F = ln(G(ξ))ξ = G'(ξ) reduces equation into the Reccati equation

$F' = q + pF + (r - 1)F\,2$ (7)

Substitute equation (4) including Eq. (7) and Eq. (5) into Eq. (3), then we will obtain polynomials in [d + G'(ξ)]j and

[d + G'(ξ)]−j ,(j = 0, 1, 2, 3, , N).

Collect each coefficient of the resulting polynomials and let it be zero. We can obtain the following twenty seven solutions [25-46]

APPLICATIONS

STO equation by (m+G'/G)

Consider the space-time fractional order STO equation

$D\alpha u(x, t) + 3\kappa(D\beta u(x, t))2 + 3\kappa uu2(x, t)D\beta u(x, t)$

t x x

$+ 3\kappa u(x, t)D2\beta u(x, t) + \kappa D3\beta u(x, t) = 0,\ t >,\ 0 < \alpha,\ \beta \le 1$ (8)

Using the transformation

$u(x, t) = U(\xi)\ \xi = \frac{\Gamma(\beta+1)}{\alpha}\mu(x^{\alpha} - lt^{\alpha})$ (9)

((8)) can be changed into an ODE equation

$l\mu u' + 3\kappa\mu^2 (u')^2 + 3\kappa\mu^2 U U' + 3\kappa\mu^2 U U'' + \kappa\mu^3 U''' = 0$ (10)

Using the balance principle we get n = 1, so eq. (4) reduces to

$U(\xi) = c-1(m + F)-1 + c0 + c1(m + F)$ (11)

After solving the algebraic equations of (10) we obtained variety of solutions as follows:

Case 01

$c_1=\mu\, d^2(r-1)-dp+q,\ c_0=c_0,\ c_1=0,$

$l=\kappa-3c_0\mu(p-2d(r-1))+3c^2+\mu^2 3d^2(r-1)^2-3dp(r-1)+p^2-qr+q$ **(12)**

Case 02

c−1 = 0, c1 = μ−μr,

l = κ−3c0μ(p − 2d(r − 1)) + 3c2 + μ2 3d2(r − 1)2 − 3dp(r − 1) + p2 −qr + q **(13)**

Case 03

c−1 = 2μ d2(r − 1) −dp + q , c1 = 0, c0 = μ(p − 2d(r − 1)),

l = κμ2 p2 − 4q(r − 1) (14)

Case 04

$c_{-1}=\mu\, d^2(r-1)-dp+q,\ c_1=\mu-\mu r,\ c_0=0,\ l=\kappa\mu^2 p^2-4q(r-1)$ (15)

Case 05

$c_{-1}=0,\ c_1=-2(\mu r-\mu),\ c_0=\mu(-(p-2d(r-1))),\ l=\kappa\mu^2 p^2-4q(r-1)$ (16)

Solution 1

(Corresponding case 01)

Type 01: When p2 − 4q(r − 1) > 0 and p(r − 1) /= 0 (orq(r − 1) =/ 0) the solution of Eqs(10)

$$u_1(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{1}{2(r-1)}[p + \sqrt{p^2 - 4q(r-1)}\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)]} \quad (17)$$

$$u_2(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{1}{2(r-1)}[p + \sqrt{p^2 - 4q(r-1)}\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)]} \quad (18)$$

$$u_3(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\ldots - \frac{1}{2(r-1)}[\ldots\sqrt{p^2 - 4q(r-1)}(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) \pm i\,\mathrm{sech}(\ldots))]} \quad (19)$$

$$u_4(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{1}{2(r-1)}[p + \sqrt{p^2 - 4q(r-1)}(\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) \pm \mathrm{cosech}(\sqrt{p^2 - 4q(r-1)}))]} \quad (20)$$

$$u_5(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\ldots - \frac{1}{4(r-1)}[2p + \sqrt{p^2 - 4q(r-1)}(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi) \pm \coth(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi))]} \quad (21)$$

$$u_6(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\ldots - \frac{1}{2(r-1)}[p + \frac{\sqrt{(A^2+B^2)(p^2-4q(r-1))} - A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi)+B}]} \quad (22)$$

$$u_7(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\ldots - \frac{1}{2(r-1)}[p + \frac{\sqrt{(B^2-A^2)(p^2-4q(r-1))} - A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi)+B}]} \quad (23)$$

Where A and B are two non-zero real constants and Satisfies B2 − A2 > 0

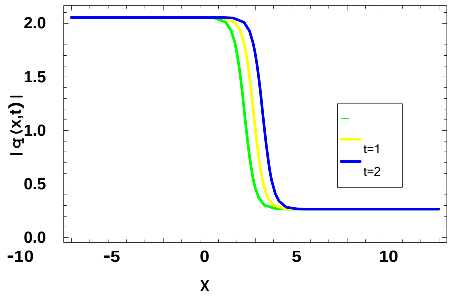

Figure 1: Solution of (17) when p = 3, r = 2, q = 1, κ = 0.08, μ = 0.8, c_0 = 2.5, d = 0.05, β = 2.

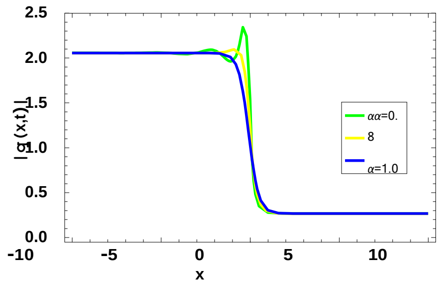

Figure 2: Solution of when (17) when p=3, r=2, q=1, κ=0.08, μ=0.8, c_0=2.5, d=0.05, β=2.

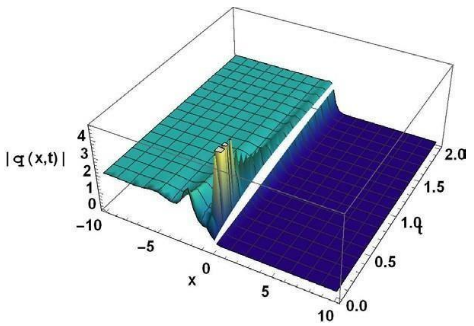

Figure 3: Solution of when (17) when p=3, r=2, q=1, κ=0.08, μ=0.8, c_0=2.5, d=0.05, β=2.

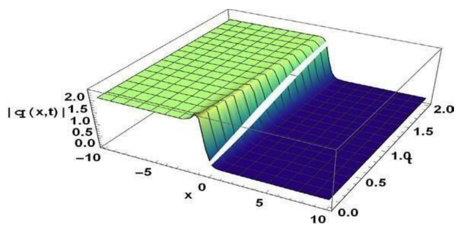

Figure 4: Solution of when (17) when p=3, r=2, q=1, κ=0.08, μ=0.8, c_0=2.5, d=0.05, β=2.

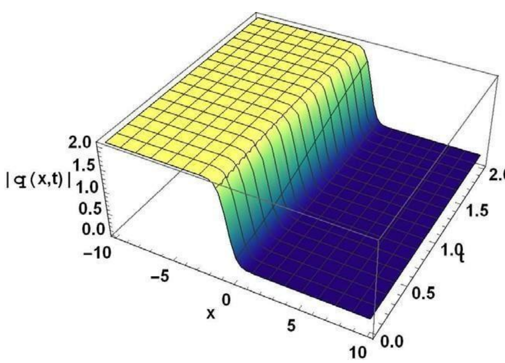

Figure 5: Solution of when (17) when p=3, r=2, q=1, κ=0.08, μ=0.8, c_0=2.5, d=0.05, β=2.

$$u_8(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\dots \frac{2q\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{\sqrt{p^2-4q(r-1)}\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) - p\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}} \quad (24)$$

$$u_9(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\dots \frac{-2q\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) - \sqrt{p^2-4q(r-1)}\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}} \quad (25)$$

$$u_{10}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{2q\cosh(\sqrt{p^2-4q(r-1)}\xi)}{p\sinh(\sqrt{p^2-4q(r-1)}\xi) - p\cosh(\sqrt{p^2-4q(r-1)}\xi) \pm i\sqrt{p^2-4q(r-1)}}} \quad (26)$$

$$u_{11}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{2q\sinh(\sqrt{p^2-4q(r-1)}\xi)}{-p\sinh(\sqrt{p^2-4q(r-1)}\xi) + \sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi) \pm \sqrt{p^2-4q(r-1)}}} \quad (27)$$

$$u_{12}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{4q\sinh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\cosh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)}{-2p\sinh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\cosh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi) + 2\sqrt{p^2-4q(r-1)}\cosh^2(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi) - \sqrt{p^2-4q(r-1)}}} \quad (28)$$

Type02: When $p^2 - 4q(r-1) < 0$ and $pr \neq 0$ $(or\, q(r-1) \neq 0)$, the Solutions of **Eqs(10)**

$$u_{13}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{1}{2r}[-p + \sqrt{4q(r-1) - p^2}\tan(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)]} \quad (29)$$

$$u_{14}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{1}{2r}[p + \sqrt{4q(r-1) - p^2}\cot(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)]} \quad (30)$$

$$u_{15}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{1}{2r}[-p + \sqrt{4q(r-1) - p^2}(\tan(\sqrt{4q(r-1)-p^2}\xi) \pm \sec(\sqrt{4q(r-1)-p^2}))]} \quad (31)$$

$$u_{16}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{1}{2r}[p + \sqrt{4q(r-1) - p^2}(\cot(\sqrt{4q(r-1)-p^2}\xi) \pm \csc(\sqrt{4q(r-1)-p^2}))]} \quad (32)$$

$$u_{17}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{1}{4r}[-2p + \sqrt{4q(r-1) - p^2}(\tan(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi) - \cot(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi))]} \quad (33)$$

$$u_{18}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{1}{2r}[-p + \frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)} - A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi) + B}]} \quad (34)$$

$$u_{19}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{1}{2r}[-p - \frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)} + \sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi) + B}]} \quad (35)$$

Where A and B are two non-zero real constants and Satisfies $A^2 - B^2 > 0$

$$u_{20}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\dots \frac{2q\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi) + p\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}} \quad (36)$$

$$u_{21}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\dots \frac{\dots}{-p\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi) + \sqrt{4q(r-1)-p^2}\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}} \quad (37)$$

$$u_{22}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{\dots}{\sqrt{4q(r-1)-p^2}\sin(\sqrt{4q(r-1)-p^2}\xi) + p\cos(\sqrt{4q(r-1)-p^2}\xi) \pm \sqrt{4q(r-1)-p^2}}} \quad (38)$$

$$u_{23}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{\dots}{-p\sin(\sqrt{4q(r-1)-p^2}\xi) + \sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi) \pm \sqrt{4q(r-1)-p^2}}} \quad (39)$$

$$u_{24}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{\dots \frac{\dots}{-2p\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi) + 2\sqrt{4q(r-1)-p^2}\cos^2(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi) - \sqrt{4q(r-1)-p^2}}} \quad (40)$$

Type 03 When $q = 0$ and $p(r-1) \neq 0$ the solutions of **Eqs(10)**

$$u_{25}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{-pd}{c_0 + \cosh(p\xi) - \sinh(p\xi)}} \quad (41)$$

$$u_{26}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d + \frac{p(\cosh(p\xi) + \sinh(p\xi))}{c_0 + \cosh(p\xi) + \sinh(p\xi)}} \quad (42)$$

Type 04 When $(r-1) \neq 0$ and $q = p = 0$ the solutions of **Eqs(10)**

$$u_{27}(x,t) = c_0 + \frac{\mu\, d^2(r-1) - dp + q}{d - \frac{1}{(r-1)\xi + c_3}} \quad (43)$$

Solution 2(Corresponding case 02)

Type 01: When $p^2 - 4q(r-1) > 0$ and $p(r-1) \neq 0$ $(or\, q(r-1) \neq 0)$, the Solutions of **Eqs(10)**

$$u_1(x,t) = c_0 - \mu(r-1)(d - \frac{1}{2(r-1)}[p + \sqrt{p^2-4q(r-1)}\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)]) \quad (44)$$

$$u_2(x,t) = c_0 - \mu(r-1)(d - \frac{1}{2(r-1)}[p + \sqrt{p^2-4q(r-1)}\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)]) \quad (45)$$

$$u_3(x,t) = c_0 - \mu(r-1)(d - \frac{1}{2(r-1)}[p + \sqrt{p^2-4q(r-1)}(\tanh(\sqrt{p^2-4q(r-1)}\xi) \pm i\,\mathrm{sech}(\sqrt{p^2-4q(r-1)}))]) \quad (46)$$

$$u_4(x,t) = c_0 - \mu(r-1)(d - \frac{1}{2(r-1)}[p + \sqrt{p^2-4q(r-1)}(\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) \pm \mathrm{cosech}(\sqrt{p^2-4q(r-1)}))]) \quad (47)$$

$$u_5(x,t) = c_0 - \mu(r-1)(d - \frac{1}{4(r-1)}[2p + \sqrt{p^2-4q(r-1)}(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi) + \coth(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi))]) \quad (48)$$

$$u_6(x,t) = c_0 - \mu(r-1)(d + \frac{1}{2(r-1)}[-p + \frac{\sqrt{(A^2+B^2)(p^2-4q(r-1))} - A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi) + B}]) \quad (49)$$

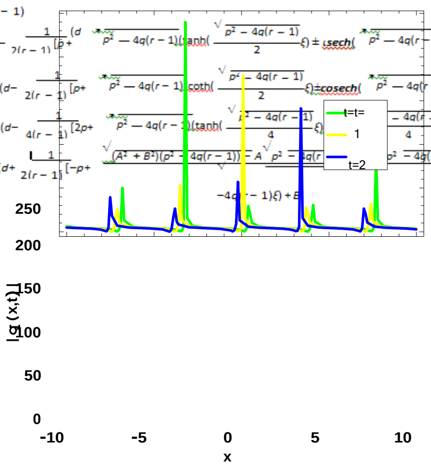

Figure 6: Solution of (29) when p=1, r=2, q=1, κ=0.06, μ=0.5, c_0=4.5, d=0.08, β=2.

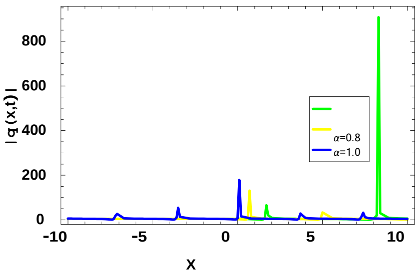

Figure 7: Solution of (29) when p=1, r=2, q=1, κ=0.06, μ=0.5, c_0=4.5, d=0.08, β=2.

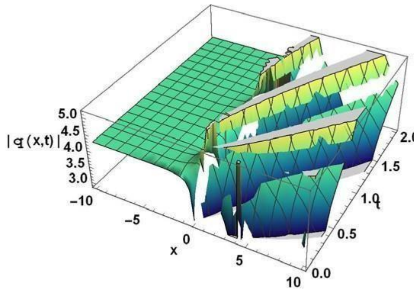

Figure 8: Solution of (29) when p=1, r=2, q=1, κ=0.06, μ=0.5, c_0 =4.5, d=0.08, β=2.

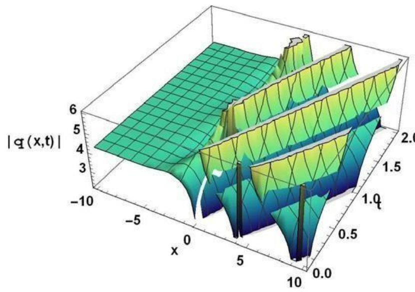

Figure 9: Solution of (29) when $p = 1$, $r= 2$, $q = 1$, $\kappa= 0.06$, $\mu = 0.5$, $c_0 = 4.5$, $d = 0.08$, $\beta= 2$.

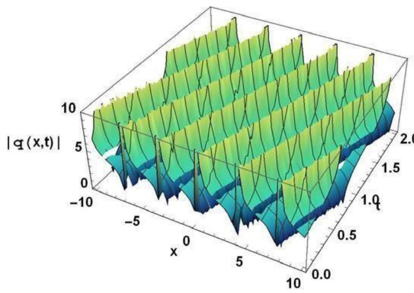

Figure10: Solution of (29) when p=1, r=2, q=1, κ=0.06, μ=0.5, c_0=4.5, d=0.08, β=2.

$$u_7(x,t) = c_0 - \mu(r-1)(d + \frac{1}{2(r-1)}[-p - \frac{\sqrt{(B^2-A^2)(p^2-4q(r-1))} - A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi) + B}]) \quad (50)$$

Where A and B are two non-zero real constants and Satisfies $B^2 - A^2 > 0$

$$u_8(x,t) = c_0 - \mu(r-1)(d + \frac{2q\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{\sqrt{p^2-4q(r-1)}\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) - p\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}) \quad (51)$$

$$u_9(x,t) = c_0 - \mu(r-1)(d + \frac{-2q\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi) - \sqrt{p^2-4q(r-1)}\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}) \quad (52)$$

$$u_{10}(x,t) = c_0 - \mu(r-1)(d + \frac{2q\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\sqrt{p^2-4q(r-1)}\xi) - p\cosh(\sqrt{p^2-4q(r-1)}\xi) \pm i\sqrt{p^2-4q(r-1)}}) \quad (53)$$

$$u_{11}(x,t) = c_0 - \mu(r-1)(d + \frac{2q\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{-p\sinh(\sqrt{p^2-4q(r-1)}\xi) + \sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi) \pm \sqrt{p^2-4q(r-1)}}) \quad (54)$$

$$u_{12}(x,t) = c_0 - \mu(r-1)(d + \frac{4q\sinh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\cosh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)}{-2p\sinh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\cosh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi) + 2\sqrt{p^2-4q(r-1)}\cosh^2(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi) - \sqrt{p^2-4q(r-1)}}) \quad (55)$$

Type02: When $p^2 - 4q(r-1) < 0$ and $pr \neq 0$ $(or\, q(r-1) \neq 0)$, the Solutions of **Eqs(10)**

$$u_{13}(x,t) = c_0 - \mu(r-1)(d + \frac{1}{2r}[-p + \sqrt{4q(r-1) - p^2}\tan(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)]) \quad (56)$$

$$u_{14}(x,t) = c_0 - \mu(r-1)(d - \frac{1}{2r}[p + \sqrt{4q(r-1) - p^2}\cot(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)]) \quad (57)$$

$$u_{15}(x,t) = c_0 - \mu(r-1)(d + \frac{1}{2r}[-p + \sqrt{4q(r-1) - p^2}(\tan(\sqrt{4q(r-1)-p^2}\xi) \pm \sec(\sqrt{4q(r-1)-p^2}))]) \quad (58)$$

$$u_{17}(x,t) = c_0 - \mu(r-1)(d + \frac{1}{4r}[-2p + \sqrt{4q(r-1) - p^2}(\tan(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi) - \cot(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi))]) \quad (60)$$

$$u_{18}(x,t) = c_0 - \mu(r-1)(d + \frac{1}{2r}[-p + \frac{\sqrt{(A^2-B^2)(4q(r-1)-p^2)} - A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi) + B}])$$

$$u_{18}(x,t)=c_0-\mu(r-1)\left(d+\frac{1}{2r}\left[-p+\frac{\sqrt{(A^2-B^2)(4q(r-1)-p^2)}-A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi)+B}\right]\right) \quad (61)$$

$$u_{19}(x,t)=c_0-\mu(r-1)\left(d+\frac{1}{2r}\left[-p-\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)}+A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi)+B}\right]\right) \quad (62)$$

Where A and B are two non-zero real constants and satisfies $A^2-B^2>0$

$$u_{20}(x,t)=c_0-\mu(r-1)\left(d-\frac{2a\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)+p\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}\right) \quad (63)$$

$$u_{21}(x,t)=c_0-\mu(r-1)\left(d+\frac{2a\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{-p\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)+\sqrt{4q(r-1)-p^2}\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}\right) \quad (64)$$

$$u_{22}(x,t)=c_0-\mu(r-1)\left(d-\frac{2a\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\sqrt{4q(r-1)-p^2}\xi)+p\cos(\sqrt{4q(r-1)-p^2}\xi)\pm\sqrt{4q(r-1)-p^2}}\right) \quad (65)$$

$$u_{23}(x,t)=c_0-\mu(r-1)\left(d+\frac{2a\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{-p\sin(\sqrt{4q(r-1)-p^2}\xi)+\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)\pm\sqrt{4q(r-1)-p^2}}\right) \quad (66)$$

$$u_{24}(x,t)=c_0-\mu(r-1)\left(d+\frac{4a\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)}{-2p\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)+2\sqrt{4q(r-1)-p^2}\cos^2(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)-\sqrt{4q(r-1)-p^2}}\right) \quad (67)$$

Type:03 When $q=0$ and $p(r-1)\neq 0$ the solutions of Eqs(10)

$$u_{25}(x,t)=c_0-\mu(r-1)\left(d+\frac{-pd}{r[d+\cosh(p\xi)-\sinh(p\xi)]}\right) \quad (68)$$

$$u_{26}(x,t)=c_0-\mu(r-1)\left(d-\frac{p[\cosh(p\xi)+\sinh(p\xi)]}{r[d+\cosh(p\xi)+\sinh(p\xi)]}\right) \quad (69)$$

Solution 04 (Corresponding case 04)

Type:04 When $(r-1)\neq 0$ and $q=p=0$ the solutions of Eqs(10)

$$u_{27}(x,t)=c_0-\mu(r-1)\left(d-\frac{1}{r\xi+c}\right) \quad (70)$$

Applications

4 Kdv-ZK equation by $(m+G'/G)$

Space-time fractional Kdv-ZK equation

consider the space-time fractional Kdv-ZK equation [19]

$$D^{\alpha}u+auu_x+u_{xxx}+c(u_{yyx}+u_{zzx})=0,\ t>,\ 0<\alpha\leq 1 \quad (71)$$

and the transformation

$$u(x,y,z,t)=U(\xi) \qquad \xi=(x+y+z)-\frac{\Gamma(\beta+1)}{\alpha}\kappa t^{\alpha} \quad (72)$$

Where κ is nonzero constant, which on substituting in (71) give us the following ODE

$$-\kappa U'+aUU'+(1+2c)U'''=0 \quad (73)$$

$$U(\xi)=c_{-1}(m+F)^{-1}+c_0+c_1(m+F)+c_{-2}(m+F)^{-2}+c_2(m+F)^2 \quad (74)$$

Case 01

$$c_{-2}=0,\ c_{-1}=0,\ c_1=-\frac{12(2c+1)(r-1)(p-2d(r-1))}{a} \quad (75)$$

$$\kappa=ac_0+(2c+1)\left(12d^2(r-1)^2-12dp(r-1)+p^2+8q(r-1)\right)$$

Solutions corresponding to Case 01 Type 01: When $p^2-4q(r-1)>0$ and $p(r-1)\neq 0$ (or $q(r-1)=0$), the Solutions of equation(73)

$$u_1(x,t)=\frac{\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\right]\right)^2}{a}+c_0 \quad (76)$$

$$u_2(x,t)=\frac{\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\right]\right)^2}{a}+c_0 \quad (77)$$

$$u_3(x,t)=\frac{\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\left(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\pm i\,\mathrm{sech}(\sqrt{p^2-4q(r-1)})\right)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\left(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\pm i\,\mathrm{sech}(\sqrt{p^2-4q(r-1)})\right)\right]\right)^2}{a}+c_0 \quad (78)$$

$$u_4(x,t)=\frac{\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\left(\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\pm \mathrm{cosech}(\sqrt{p^2-4q(r-1)})\right)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{2(r-1)}\left[p+\sqrt{p^2-4q(r-1)}\left(\coth(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)\pm \mathrm{cosech}(\sqrt{p^2-4q(r-1)})\right)\right]\right)^2}{a}+c_0 \quad (79)$$

$$u_5(x,t)=\frac{\left(d-\frac{1}{4(r-1)}\left[2p+\sqrt{p^2-4q(r-1)}\left(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\pm\coth(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\right)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{4(r-1)}\left[2p+\sqrt{p^2-4q(r-1)}\left(\tanh(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\pm\coth(\frac{\sqrt{p^2-4q(r-1)}}{4}\xi)\right)\right]\right)^2}{a}+c_0 \quad (80)$$

$$u_6(x,t)=\frac{\left(d+\frac{1}{2(r-1)}\left[-p+\frac{\sqrt{(A^2+B^2)(p^2-4q(r-1))}-A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi)+B}\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{2(r-1)}\left[-p+\frac{\sqrt{(A^2+B^2)(p^2-4q(r-1))}-A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi)+B}\right]\right)^2}{a}+c_0 \quad (81)$$

$$u_7(x,t)=\frac{\left(d+\frac{1}{2(r-1)}\left[-p-\frac{\sqrt{(B^2-A^2)(p^2-4q(r-1))}-A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi)+B}\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{2(r-1)}\left[-p-\frac{\sqrt{(B^2-A^2)(p^2-4q(r-1))}-A\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)}{A\sinh(\sqrt{p^2-4q(r-1)}\xi)+B}\right]\right)^2}{a}+c_0$$

Where A and B are two non-zero real constants and Satisfies $B^2-A^2>0$ (82)

$$u_8(x,t)=\frac{\left(d+\frac{-2a\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{\sqrt{p^2-4q(r-1)}\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)-p\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2(d+\varphi(\xi))^2}{a}+c_0 \quad (83)$$

$$u_9(x,t)=\frac{\left(d+\frac{-2a\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)-\sqrt{p^2-4q(r-1)}\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{-2a\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)-\sqrt{p^2-4q(r-1)}\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}\right)^2}{a}+c_0 \quad (84)$$

$$u_{10}(x,t)=\frac{\left(d+\frac{2a\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\sqrt{p^2-4q(r-1)}\xi)-p\cosh(\sqrt{p^2-4q(r-1)}\xi)\pm i\sqrt{p^2-4q(r-1)}}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{2a\cosh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{p\sinh(\sqrt{p^2-4q(r-1)}\xi)-p\cosh(\sqrt{p^2-4q(r-1)}\xi)\pm i\sqrt{p^2-4q(r-1)}}\right)^2}{a}+c_0 \quad (85)$$

$$u_{11}(x,t)=\frac{\left(d+\frac{2a\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{-p\sinh(\sqrt{p^2-4q(r-1)}\xi)+\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)\pm\sqrt{p^2-4q(r-1)}}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{2a\sinh(\frac{\sqrt{p^2-4q(r-1)}}{2}\xi)}{-p\sinh(\sqrt{p^2-4q(r-1)}\xi)+\sqrt{p^2-4q(r-1)}\cosh(\sqrt{p^2-4q(r-1)}\xi)\pm\sqrt{p^2-4q(r-1)}}\right)^2}{a}+c_0 \quad (86)$$

$$u_{12}(x,t)=\frac{(d+\varphi(\xi))(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2(d+\varphi(\xi))^2}{a}+c_0 \quad (87)$$

Type02: When $p^2-4q(r-1)<0$ and $pr\neq 0$ (or $q(r-1)\neq 0$), the Solutions of equation(73)

$$u_{13}(x,t)=\frac{\left(d+\frac{1}{2r}\left[-p+\sqrt{4q(r-1)-p^2}\tan(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{2r}\left[-p+\sqrt{4q(r-1)-p^2}\tan(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)\right]\right)^2}{a}+c_0 \quad (88)$$

$$u_{14}(x,t)=\frac{\left(d-\frac{1}{2r}\left[p+\sqrt{4q(r-1)-p^2}\cot(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{2r}\left[p+\sqrt{4q(r-1)-p^2}\cot(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)\right]\right)^2}{a}+c_0 \quad (89)$$

$$u_{15}(x,t)=\frac{\left(d+\frac{1}{2r}\left[-p+\sqrt{4q(r-1)-p^2}\left(\tan(\sqrt{4q(r-1)-p^2}\xi)\pm\sec(\sqrt{4q(r-1)-p^2})\right)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{2r}\left[-p+\sqrt{4q(r-1)-p^2}\left(\tan(\sqrt{4q(r-1)-p^2}\xi)\pm\sec(\sqrt{4q(r-1)-p^2})\right)\right]\right)^2}{a}+c_0 \quad (90)$$

$$u_{16}(x,t)=\frac{\left(d-\frac{1}{2r}\left[p+\sqrt{4q(r-1)-p^2}\left(\cot(\sqrt{4q(r-1)-p^2}\xi)\pm\csc(\sqrt{4q(r-1)-p^2})\right)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{2r}\left[p+\sqrt{4q(r-1)-p^2}\left(\cot(\sqrt{4q(r-1)-p^2}\xi)\pm\csc(\sqrt{4q(r-1)-p^2})\right)\right]\right)^2}{a}+c_0 \quad (91)$$

$$u_{17}(x,t)=\frac{\left(d+\frac{1}{4r}\left[-2p+\sqrt{4q(r-1)-p^2}\left(\tan(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)-\coth(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\right)\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{4r}\left[-2p+\sqrt{4q(r-1)-p^2}\left(\tan(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)-\coth(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\right)\right]\right)^2}{a}+c_0 \quad (92)$$

$$u_{18}(x,t)=\frac{\left(d+\frac{1}{2r}\left[-p+\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)}-A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi)+B}\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{2r}\left[-p+\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)}-A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi)+B}\right]\right)^2}{a}+c_0 \quad (93)$$

$$u_{19}(x,t)=\frac{\left(d+\frac{1}{2r}\left[-p-\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)}+A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi)+B}\right]\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{1}{2r}\left[-p-\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-p^2)}+A\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)}{A\sin(\sqrt{4q(r-1)-p^2}\xi)+B}\right]\right)^2}{a}+c_0$$

Where A and B are two non-zero real constants and Satisfies $A^2-B^2>0$ (94)

$$u_{20}(x,t)=\frac{\left(d-\frac{2a\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)+p\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{2a\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)+p\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}\right)^2}{a}+c_0 \quad (95)$$

$$u_{21}(x,t)=\frac{\left(d+\frac{2a\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{-p\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)+\sqrt{4q(r-1)-p^2}\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{2a\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{-p\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)+\sqrt{4q(r-1)-p^2}\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}\right)^2}{a}+c_0 \quad (96)$$

$$u_{22}(x,t)=\frac{\left(d-\frac{2a\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\sqrt{4q(r-1)-p^2}\xi)+p\cos(\sqrt{4q(r-1)-p^2}\xi)\pm\sqrt{4q(r-1)-p^2}}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{2a\cos(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{\sqrt{4q(r-1)-p^2}\sin(\sqrt{4q(r-1)-p^2}\xi)+p\cos(\sqrt{4q(r-1)-p^2}\xi)\pm\sqrt{4q(r-1)-p^2}}\right)^2}{a}+c_0 \quad (97)$$

$$u_{23}(x,t)=\frac{\left(d+\frac{2a\sin(\frac{\sqrt{4q(r-1)-p^2}}{2}\xi)}{-p\sin(\sqrt{4q(r-1)-p^2}\xi)+\sqrt{4q(r-1)-p^2}\cos(\sqrt{4q(r-1)-p^2}\xi)\pm\sqrt{4q(r-1)-p^2}}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2(d+\varphi(\xi))^2}{a}+c_0 \quad (98)$$

$$u_{24}(x,t)=\frac{\left(d+\frac{4a\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)}{-2p\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)+2\sqrt{4q(r-1)-p^2}\cos^2(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)-\sqrt{4q(r-1)-p^2}}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{4a\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)}{-2p\sin(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)\cos(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)+2\sqrt{4q(r-1)-p^2}\cos^2(\frac{\sqrt{4q(r-1)-p^2}}{4}\xi)-\sqrt{4q(r-1)-p^2}}\right)^2}{a}+c_0 \quad (99)$$

Type:03 When $q=0$ and $p(r-1)\neq 0$ the solutions of equation(73)

$$u_{25}(x,t)=\frac{\left(d+\frac{-pd}{r[d+\cosh(p\xi)-\sinh(p\xi)]}\right)(12(2c+1)(r-1)(2dr-2d-p))}{a}-\frac{12(2c+1)(r-1)^2\left(d+\frac{-pd}{r[d+\cosh(p\xi)-\sinh(p\xi)]}\right)^2}{a}+c_0 \quad (100)$$

$$u_{28}(x,t)=\frac{\left(d-\frac{\rho(\cosh(\rho\xi)+\sinh(\rho\xi))}{c(d+\cosh(\rho\xi)+\sinh(\rho\xi))}\right)(12(2c+1)(r-1)(2dr-2d-\rho))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{\rho(\cosh(\rho\xi)+\sinh(\rho\xi))}{c(d+\cosh(\rho\xi)+\sinh(\rho\xi))}\right)^2}{a}+c_0 \quad (101)$$

Solution 04 (Corresponding case 04)

Type:04 When $(r-1)\neq 0$ and $q=\rho=0$ the solutions of equation (73)

$$u_{29}(x,t)=\frac{\left(d-\frac{1}{c\xi+c_1}\right)(12(2c+1)(r-1)(2dr-2d-\rho))}{a}-\frac{12(2c+1)(r-1)^2\left(d-\frac{1}{c\xi+c_1}\right)^2}{a}+c_0 \quad (102)$$

Case 02

$$c_{-2}=-\frac{12(2c+1)(d^2(r-1)-d\rho+q)^2}{a},\ c_{-1}=\frac{12(2c+1)(2d(r-1)-\rho)(d^2(r-1)-d\rho+q)}{a} \quad (103)$$

$c_1=0,\ c_2=0,\ \kappa=ac_0+(2c+1)\,12d^2(r-1)^2-12d\rho(r-1)+\rho^2+8q(r-1)$

Type 01: When $\rho^2-4q(r-1)>0$ and $\rho(r-1)\neq 0$ (or $q(r-1)\neq 0$), the Solutions of equation (73)

$$u_1(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\tanh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\tanh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\right]\right)^2}+c_0 \quad (104)$$

$$u_2(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\coth\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\coth\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\right]\right)^2}+c_0 \quad (105)$$

$$u_3(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\left(\tanh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{\rho^2-4q(r-1)}\right)\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\left(\tanh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{\rho^2-4q(r-1)}\right)\right)\right]\right)^2}+c_0 \quad (106)$$

$$u_4(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\left(\coth\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\pm \mathrm{csch}\left(\sqrt{\rho^2-4q(r-1)}\right)\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\sqrt{\rho^2-4q(r-1)}\left(\coth\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)\pm \mathrm{csch}\left(\sqrt{\rho^2-4q(r-1)}\right)\right)\right]\right)^2}+c_0 \quad (107)$$

$$u_5(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{4(r-1)}\left[2\rho+\sqrt{\rho^2-4q(r-1)}\left(\tanh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\pm\coth\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{4(r-1)}\left[2\rho+\sqrt{\rho^2-4q(r-1)}\left(\tanh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\pm\coth\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\right)\right]\right)^2}+c_0 \quad (108)$$

$$u_6(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\frac{\sqrt{(A^2+B^2)(\rho^2-4q(r-1))}-A\sqrt{\rho^2-4q(r-1)}\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{A\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)+B}\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2(r-1)}\left[\rho+\frac{\sqrt{(A^2+B^2)(\rho^2-4q(r-1))}-A\sqrt{\rho^2-4q(r-1)}\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{A\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)+B}\right]\right)^2}+c_0 \quad (109)$$

$$u_7(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{1}{2(r-1)}\left[-\rho-\frac{\sqrt{(B^2-A^2)(\rho^2-4q(r-1))}-A\sqrt{\rho^2-4q(r-1)}\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{A\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)+B}\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{1}{2(r-1)}\left[-\rho-\frac{\sqrt{(B^2-A^2)(\rho^2-4q(r-1))}-A\sqrt{\rho^2-4q(r-1)}\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{A\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)+B}\right]\right)^2}+c_0$$

Where A and B are two non-zero real constants and Satisfies $B^2-A^2>0$ (110)

$$u_8(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{2q\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}{\sqrt{\rho^2-4q(r-1)}\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)-\rho\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{2q\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}{\sqrt{\rho^2-4q(r-1)}\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)-\rho\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}\right)^2}+c_0 \quad (111)$$

$$u_9(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{-2q\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}{\rho\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)-\sqrt{\rho^2-4q(r-1)}\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{-2q\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}{\rho\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)-\sqrt{\rho^2-4q(r-1)}\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{2}\xi\right)}\right)^2}+c_0 \quad (112)$$

$$u_{10}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{2q\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{\rho\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)-\rho\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)\pm i\sqrt{\rho^2-4q(r-1)}}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{2q\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{\rho\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)-\rho\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)\pm i\sqrt{\rho^2-4q(r-1)}}\right)^2}+c_0 \quad (113)$$

$$u_{11}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{2q\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{-\rho\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)+\sqrt{\rho^2-4q(r-1)}\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)\pm\sqrt{\rho^2-4q(r-1)}}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{2q\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)}{-\rho\sinh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)+\sqrt{\rho^2-4q(r-1)}\cosh\left(\sqrt{\rho^2-4q(r-1)}\xi\right)\pm\sqrt{\rho^2-4q(r-1)}}\right)^2}+c_0 \quad (114)$$

$$u_{12}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{4q\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)}{-2\rho\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)+2\sqrt{\rho^2-4q(r-1)}\cosh^2\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)-\sqrt{\rho^2-4q(r-1)}}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{4q\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)}{-2\rho\sinh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)\cosh\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)+2\sqrt{\rho^2-4q(r-1)}\cosh^2\left(\frac{\sqrt{\rho^2-4q(r-1)}}{4}\xi\right)-\sqrt{\rho^2-4q(r-1)}}\right)^2}+c_0 \quad (115)$$

Type02: When $\rho^2-4q(r-1)<0$ and $\rho r\neq 0$ (or $q(r-1)\neq 0$), the Solutions of equation (73)

$$u_{13}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{1}{2r}\left[-\rho+\sqrt{4q(r-1)-\rho^2}\tan\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{1}{2r}\left[-\rho+\sqrt{4q(r-1)-\rho^2}\tan\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)\right]\right)^2}+c_0 \quad (116)$$

$$u_{14}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2r}\left[\rho+\sqrt{4q(r-1)-\rho^2}\cot\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2r}\left[\rho+\sqrt{4q(r-1)-\rho^2}\cot\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)\right]\right)^2}+c_0 \quad (117)$$

$$u_{15}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{1}{2r}\left[-\rho+\sqrt{4q(r-1)-\rho^2}\left(\tan\left(\sqrt{4q(r-1)-\rho^2}\xi\right)\pm\sec\left(\sqrt{4q(r-1)-\rho^2}\right)\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{1}{2r}\left[-\rho+\sqrt{4q(r-1)-\rho^2}\left(\tan\left(\sqrt{4q(r-1)-\rho^2}\xi\right)\pm\sec\left(\sqrt{4q(r-1)-\rho^2}\right)\right)\right]\right)^2}+c_0 \quad (118)$$

$$u_{16}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{2r}\left[\rho+\sqrt{4q(r-1)-\rho^2}\left(\cot\left(\sqrt{4q(r-1)-\rho^2}\xi\right)\pm\csc\left(\sqrt{4q(r-1)-\rho^2}\right)\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{2r}\left[\rho+\sqrt{4q(r-1)-\rho^2}\left(\cot\left(\sqrt{4q(r-1)-\rho^2}\xi\right)\pm\csc\left(\sqrt{4q(r-1)-\rho^2}\right)\right)\right]\right)^2}+c_0 \quad (119)$$

$$u_{17}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{1}{4r}\left[-2\rho+\sqrt{4q(r-1)-\rho^2}\left(\tan\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)-\cot\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)\right)\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{1}{4r}\left[-2\rho+\sqrt{4q(r-1)-\rho^2}\left(\tan\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)-\cot\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)\right)\right]\right)^2}+c_0 \quad (120)$$

$$u_{18}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{1}{2r}\left[-\rho+\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-\rho^2)}-A\sqrt{4q(r-1)-\rho^2}\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)}{A\sin\left(\sqrt{4q(r-1)-\rho^2}\xi\right)+B}\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{1}{2r}\left[-\rho+\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-\rho^2)}-A\sqrt{4q(r-1)-\rho^2}\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)}{A\sin\left(\sqrt{4q(r-1)-\rho^2}\xi\right)+B}\right]\right)^2}+c_0 \quad (121)$$

$$u_{19}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{1}{2r}\left[-\rho-\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-\rho^2)}+A\sqrt{4q(r-1)-\rho^2}\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)}{A\sin\left(\sqrt{4q(r-1)-\rho^2}\xi\right)+B}\right]\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{1}{2r}\left[-\rho-\frac{\pm\sqrt{(A^2-B^2)(4q(r-1)-\rho^2)}+A\sqrt{4q(r-1)-\rho^2}\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)}{A\sin\left(\sqrt{4q(r-1)-\rho^2}\xi\right)+B}\right]\right)^2}+c_0 \quad (122)$$

Where A and B are two non-zero real constants and Satisfies $A^2-B^2>0$

$$u_{20}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{2q\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}{\sqrt{4q(r-1)-\rho^2}\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)+\rho\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{2q\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}{\sqrt{4q(r-1)-\rho^2}\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)+\rho\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}\right)^2}+c_0 \quad (123)$$

$$u_{21}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{2q\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}{-\rho\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)+\sqrt{4q(r-1)-\rho^2}\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{2q\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}{-\rho\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)+\sqrt{4q(r-1)-\rho^2}\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{2}\xi\right)}\right)^2}+c_0 \quad (124)$$

$$u_{22}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{2q\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)}{\sqrt{4q(r-1)-\rho^2}\sin\left(\sqrt{4q(r-1)-\rho^2}\xi\right)+\rho\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)\pm\sqrt{4q(r-1)-\rho^2}}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{2q\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)}{\sqrt{4q(r-1)-\rho^2}\sin\left(\sqrt{4q(r-1)-\rho^2}\xi\right)+\rho\cos\left(\sqrt{4q(r-1)-\rho^2}\xi\right)\pm\sqrt{4q(r-1)-\rho^2}}\right)^2}+c_0 \quad (125)$$

$$u_{24}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{4q\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)}{-2\rho\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)+2\sqrt{4q(r-1)-\rho^2}\cos^2\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)-\sqrt{4q(r-1)-\rho^2}}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{4q\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)}{-2\rho\sin\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)\cos\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)+2\sqrt{4q(r-1)-\rho^2}\cos^2\left(\frac{\sqrt{4q(r-1)-\rho^2}}{4}\xi\right)-\sqrt{4q(r-1)-\rho^2}}\right)^2}+c_0 \quad (127)$$

Type:03 When $q=0$ and $\rho(r-1)\neq 0$ the solutions of equation (73)

$$u_{25}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d+\frac{-\rho d}{c(d+\cosh(\rho\xi)-\sinh(\rho\xi))}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d+\frac{-\rho d}{c(d+\cosh(\rho\xi)-\sinh(\rho\xi))}\right)^2}+c_0 \quad (128)$$

$$u_{26}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{\rho(\cosh(\rho\xi)+\sinh(\rho\xi))}{c(d+\cosh(\rho\xi)+\sinh(\rho\xi))}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{\rho(\cosh(\rho\xi)+\sinh(\rho\xi))}{c(d+\cosh(\rho\xi)+\sinh(\rho\xi))}\right)^2}+c_0 \quad (129)$$

Solution 04 (Corresponding case 04)

Type:04 When $(r-1)\neq 0$ and $q=\rho=0$ the solutions of equation (73)

$$u_{27}(x,t)=\frac{12(2c+1)(2dr-2d-\rho)(d^2r-d^2-d\rho+q)}{a\left(d-\frac{1}{c\xi+c_1}\right)}-\frac{12(2c+1)(d^2r-d^2-d\rho+q)^2}{a\left(d-\frac{1}{c\xi+c_1}\right)^2}+c_0 \quad (130)$$

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